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A peculiar Maxwell's Demon observed in a time-dependent stadium-like billiard

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ABSTRACT

The dynamics of a driven stadium-like billiard is considered using the formalism of discrete mappings. The model presents a resonant velocity that depends on the rotation number around fixed points and external boundary perturbation which plays an important separation rule in the model. We show that particles exhibiting Fermi acceleration (initial velocity is above the resonant one) are scaling invariant with respect to the initial velocity and external perturbation. However, initial velocities below the resonant one lead the particles to decelerate therefore unlimited energy growth is not observed. This phenomenon may be interpreted as a specific Maxwell's Demon which may separate fast and slow billiard particles.

1. Introduction

As is known, typical Hamiltonian systems are non-integrable and non-ergodic [1]. The phase space of such systems is divided into regions with regular and chaotic dynamics. Such a division leads to the stickiness phenomenon [2,3] which is manifested through the fact that a phase trajectory in a chaotic region passing near enough a Kolmogorov–Arnold–Moser (KAM) island, evolves there almost regularly during a very long period of time. However, when an orbit resides in a chaotic region far from the set of KAM regions, it moves chaotically in the sense two nearby initial conditions apart from each other exponentially as the time evolves. Therefore the stickiness of phase trajectories has a crucial influence on the transport properties of Hamiltonian systems, and its relation to physical systems is one of the most important open problems of nonlinear dynamics [4].

Billiards are Hamiltonian systems with divided phase space, and the stickiness phenomenon is inherent in such dynamical systems. For a family of billiard systems such as mushrooms [5] the coexistence of ergodic components with stickiness has been shown [4]. The notion of billiard in the contemporary sense is known since Birkhoff [6] who studied a problem of the free motion of a point particle on a manifold. Than latter investigations related to the mixing property in hard ball systems were done by Krylov [7] while Sinai [8] started the rigorous investigation of such systems.

A billiard problem consists of the motion of a point-like particle moving inside and suffering collisions with a region with a piecewise-smooth boundary [9]. If the boundary in the collision point is smooth then the billiard ball reflects from it such that the component of the normal velocity changes its sign but the tangent one remains constant. Therefore, in Euclidean space, the incidence angle is equal to the angle of reflection, and this defines a specular reflection. Billiard systems became one of the most active and popular research areas of physics and mathematics providing a fertile source of new ideas in mechanics,

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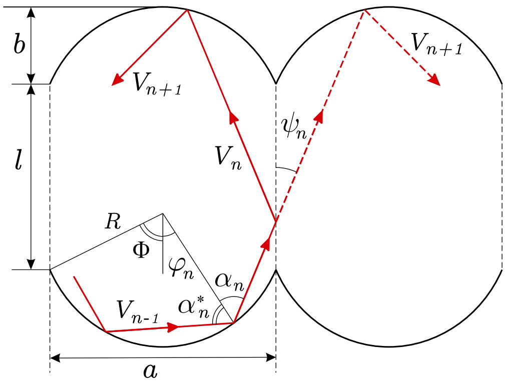

Fig. 1. (Color online). A stadium-like billiard and its geometric parameters [18].

statistical physics, astrophysics and other fields of natural sciences. In particular, applications of billiard problems can be found in quantum dots [10], optics [11], microwaves [12], ultra-cold atoms trapped in a laser potential [13,14], micro gravity and billiard lasers [15,16].

If the billiard boundary is perturbed in time, depending on the geometrical properties of the boundary and on the initial conditions, the particle can accumulate energy along its trajectory leading to a phenomenon called as Fermi acceleration (FA). Introduced in 1949 by Fermi [17], FA consists in the unlimited energy growth of a classical particle which is suffering elastic collisions with infinitely heavy and moving obstacles. The conditions of the existence of FA in time-dependent billiards are the presence of a chaotic component [18], hyperbolic fixed points [19], or heteroclinic orbits [20,21] in phase space of the corresponding billiards with the fixed boundary. Also, in Ref. [22] it was advanced a conjecture that pseudointegrability is a sufficient condition for achieving unbounded energy growth in billiards with smoothly perturbed boundaries. It should be noted however that the dynamics in the presence of dissipation, as for example the case of inelastic collisions or drag force, the phenomenon of FA is suppressed [21,23–25].

Recently on the basis of the analysis of nearly integrable focusing billiards it has been found that periodic boundary perturbations may lead not only to acceleration but also to retardation of particles and, as a consequence, to separation of particles by their velocities [26]. This significant phenomenon may be treated as a specific Maxwell's Demon, when a weak boundary perturbation leads to the formation of two ensembles of slow and fast particles [27].

In this paper we study the behavior of the average velocity of the particle in a driven stadium-like billiard. We show that depending on the initial velocity, the particles can either accelerate or decelerate. The separation of such regimes is characterized by a resonance velocity that depends on the control parameters. When the particle is exhibiting unlimited energy growth, such a phenomenon is shown to be scaling invariant with respect to the initial velocity and external boundary perturbation.

The paper is organized as follows. In Section 2 we present the model, the mapping and the dynamical variables involved in the description of the problem. Section 3 is devoted to discuss our results for the resonance, the phenomenon of Fermi acceleration and the decay of energy. Final remarks and conclusions are drawn in Section 4.

2. The model and its dynamical variables

In our analysis we follow a simplified approach proposed in Refs. [18,26] in which a family of two-dimensional stadium-like billiards is described by a nonlinear map. It is known [26] that in a nearly integrable stadium there is a critical velocity V_r which corresponds to a resonance between boundary periodic perturbations and the motion nearby KAM islands. Considering initial velocities above this critical value, FA is observed, while for initial velocities lower than V_r, decreasing of the average velocity in the particle ensemble takes place.

Consider a nearly integrable stadium with periodically moving focusing components under the law $B(t) = B_0 \cos \omega t$, where ω is the frequency and B_0 is the amplitude of boundary perturbations. Assume that the depth of the focusing components is sufficiently small, $b \ll a$, and these components have radius $R = (a^2 + 4b^2)/(8b)$ with the angle measured 2Φ, where $\Phi = \arcsin(a/(2R))$. Figs. 1 and 2 show a schematic presentation of the model and its dynamical variables. As shown in Refs. [28,29], the stochasticity condition for such a billiard family is $l/2R \approx 4bl/a^2 > 1$.

To analyze the model, two cases of the particle motion have to be considered [26]: (i) the particle undergoes a series of successive collisions with the same focusing component. This happens when $\Phi \geq |\varphi_{n+1}|$. (ii) After the collision with one focusing component the particle collides with the opposite focusing component. This takes place if $\Phi < |\varphi_{n+1}|$. For both cases, the expressions for the particle velocity V_n and the angle α_n are given by

$$\alpha_n = \arcsin\left(V_n/V_{n+1} \sin \alpha^*_n\right),$$
$$V_{n+1} = \sqrt{{V_n}^2 + 4B_n^2 + 4V_nB_n \cos \alpha^*_n},$$

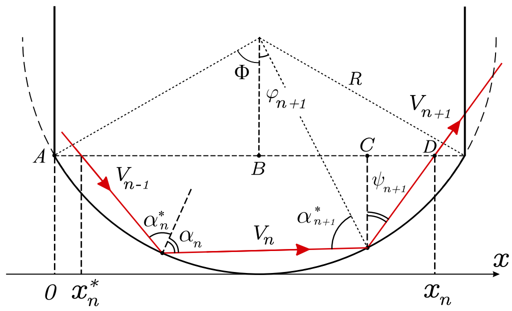

Fig. 2. (Color online). Dynamical variables in a time-dependent billiard [26].

where B_n is the velocity of the boundary at the collision time t_n.

From the geometrical analysis we obtain for case (i) the map has the following form [26]:

$$
\begin{aligned}
\alpha^*_{n+1} &= \alpha_n,\\
\varphi_{n+1} &= \varphi_n + \pi - 2\alpha_n \pmod{2\pi},\\
t_{n+1} &= t_n + \frac{2R\cos\alpha_n}{V_n}.
\end{aligned}
$$

Now, taking into account that the angle α^*_n is changed in the opposite direction than α_n and the angle φ_n should have the reversed sign, one can derive the billiard map for the case (ii) [26]:

$$
\begin{aligned}
\psi_n &= \alpha_n - \varphi_n,\\
\alpha^*_{n+1} &= \arcsin\left[\sin(\psi_n + \Phi) - \frac{x^*_{n+1}}{R}\cos\psi_n\right],\\
\varphi_{n+1} &= \psi_n - \alpha^*_{n+1},\\
t_{n+1} &= t_n + \frac{R(\cos\varphi_n + \cos\varphi_{n+1} - 2\cos\Phi) + l}{V_n\cos\psi_n},\\
x_n &= \frac{R}{\cos\psi_n}[\sin\alpha_n + \sin(\Phi - \psi_n)],\\
x^*_{n+1} &= x_n + l\tan\psi_n \pmod{a}.
\end{aligned}
$$

3. Numerical results: resonance, Fermi acceleration, scaling and decay of energy

The linearization of the unperturbed map (i.e. at $B_0 = 0$), as obtained in Refs. [26,29], produces a rotation number

$$\sigma_m = \arccos\left(1 - (8bl)/(a^2\cos^2\psi_m)\right),$$

where $\psi_m = \arctan(ma/l)$, $m \geq 0$, are the fixed points corresponding to collisions of billiard particles with the center of the focusing component. For $m = 0$ the particle moves vertically; for $m = 1$ the particle shifts into one billiard table in the billiard unfolding, and so on. The time between two collisions is approximately given by $\tau_m \approx l/(V\cos\psi_m)$. Thus, the rotation period is $T_{\rm rot} = 2\pi\tau_m/\sigma_m$. If $T_{\rm rot}$ is equal to the period of the external perturbation, a resonance between rotation around a fixed point and boundary oscillations is observed. The resonant velocity V_r is given by Loskutov and Ryabov [26]

$$V_r = \frac{\omega l}{\cos\psi_m \arccos(1 - 8bl/(a\cos\psi_m)^2)}. \tag{1}$$

It is also good to remember that such resonance condition only happens when the defocusing mechanism is not active. It was recently shown [29], that the control parameters a, b and l are scaling invariant for this transition with respect to the maximum Lyapunov exponent. In that way, we have fixed these control parameters as $a = 0.5$, $b = 0.01$, $l = 1$ for the whole numerical analysis. Also, we considered $m = 1$, which means that we are interested in the resonance of the larger islands (period-1) in the phase space, which seem to be more influenced than the other islands (period-2 and period-3). This combination of parameters yields a resonant velocity $V_r = 1.2$.

Let us discuss the behavior of the average velocity for an initial velocity above V_r, i.e. $V_0 > V_r$. The average was taken along the orbit as

$$V_i = \frac{1}{n}\sum_{j=1}^{n} V_j, \tag{2}$$

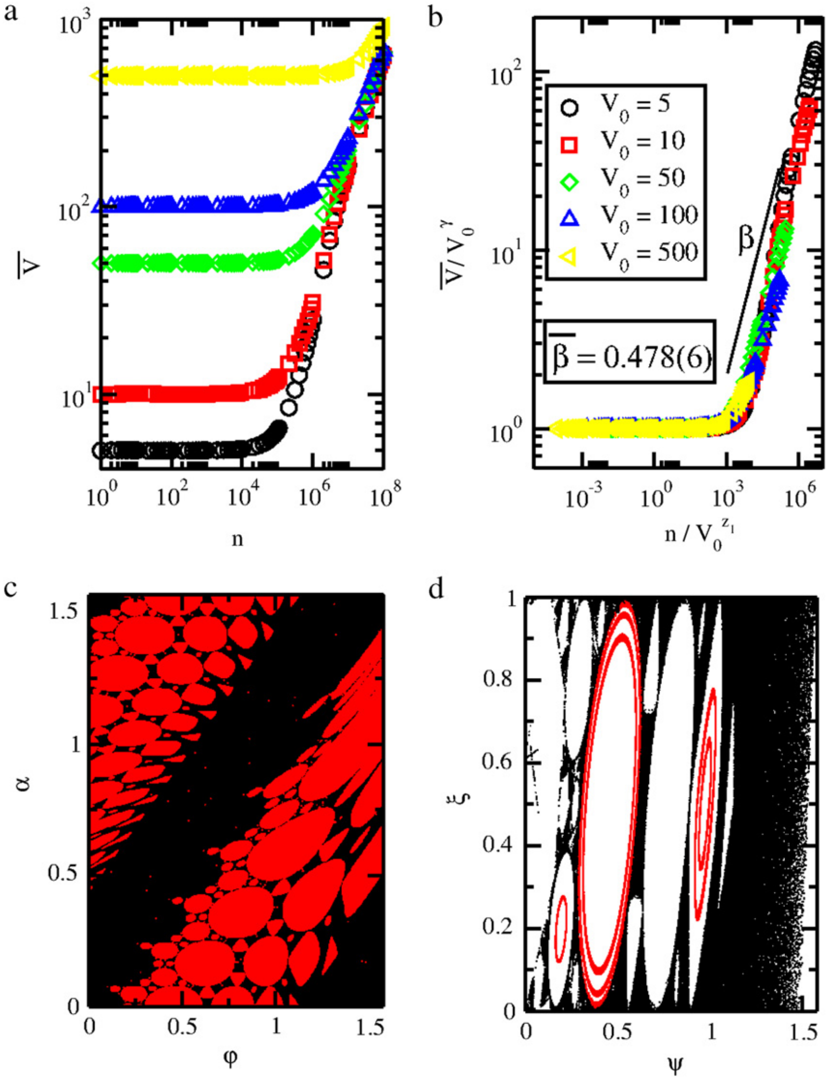

Fig. 3. (Color online). (a) Behavior of $\overline{V}$ vs n for $V_0 > V_r$. (b) Overlap of all curves shown in (a) onto a single plot. (c) Grid of 500×500 initial conditions ($\alpha, \varphi \in [0, \pi/2]$) for $V_0 = 5$. (d) Plot of ξ vs ψ for different initial conditions. In all figures, the billiard parameters are: $a = 0.5$, $b = 0.01$, $l = 1$, $u_0 = 0.01$, yielding a resonant velocity $V_r = 1.2$.

and along an ensemble of M different initial conditions

$$\overline{V} = \frac{1}{M} \sum_{i=1}^{M} V_i. \tag{3}$$

We evolved the dynamics considering two distinct cases: (i) ranging the initial velocity ($V_0 > V_r$) and keeping fixed the amplitude of external boundary perturbation, and (ii) keeping fixed the initial velocity and ranging the amplitude of the time external perturbation of the boundary. Let us first discuss the case (i).

Fig. 3(a) shows the behavior of $\overline{V}$ as a function of the number of collisions n for initial velocities obtained for an ensemble of $M = 5000$ initial conditions (α, φ) chosen in a region where FA is observed (black regions of Fig. 3(c)) for the billiard parameters mentioned previously. Each initial condition was evolved in time up to 10^8 collisions. One can see that, at first, the average velocity stays in a constant plateau nearby the initial value V_0, $\overline{V} \propto V_0^{\gamma}$ with $\gamma = 1$. Then the velocity curve bends towards a regime of growth characterized as $\overline{V} \propto n^{\beta}$, where $\beta \approx 0.5$. The changeover from constant to the regime of growth is marked by a crossover n_x that is written as $n_x \propto V_0^{z_1}$ with $z_1 = 1.85(3)$. Fig. 3(b) shows a rescaling of the axes yielding an overlap of different curves, corresponding to different initial velocities, onto a single plot revealing a scaling invariance of $\overline{V}$ with respect to V_0. Thus, for the initial velocity above the resonant one V_r, FA is observed.

Fig. 3(c) shows the phase plane (α, φ). Initial conditions that lead to FA are marked by black color. These black regions correspond to chaotic regions in the billiard map. Red parts in this figure symbolize KAM islands and, in our simulations,

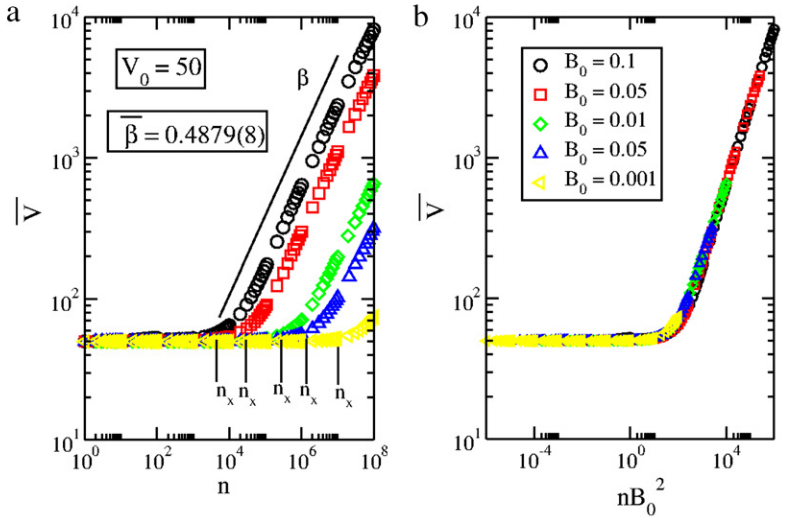

Fig. 4. (Color online) (a) Behavior of $\overline{V}$ vs n for $V_0 = 50$ and different values of the external perturbation amplitude B_0. (b) Overlap of the curves shown in (a) onto a single plot.

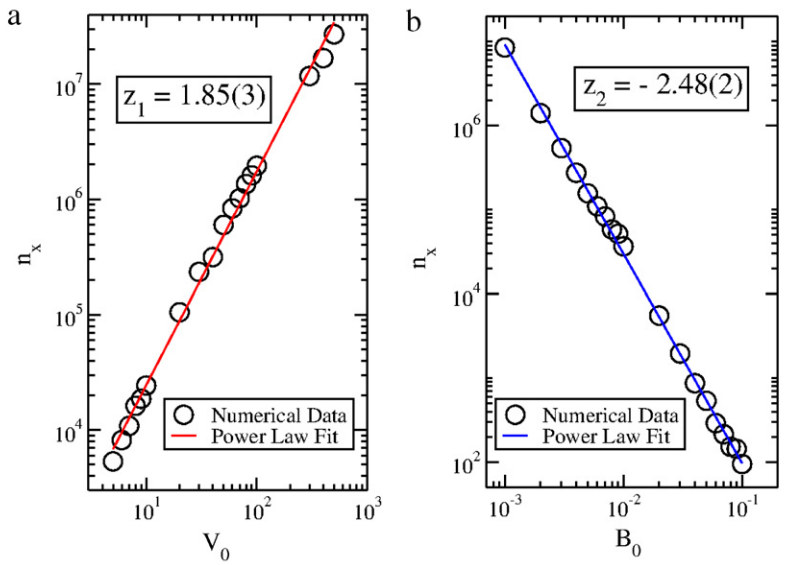

Fig. 5. (Color online) Plot of: (a) $V_0 vs. n_x$ where a power law fitting furnishes $z_1 = 1.85(3)$; (b) $B_0 vs. n_x$ with a slope of $z_2 = -2.48(2)$.

initial conditions belonging to these regions do not show the particles to accumulate energy. We have evolved this set of initial conditions up to 10^8 collisions and still FA was not observed. Some of the regular regions shown in red in Fig. 3(c) are plotted as red in Fig. 3(d), where $\xi = 1/2 + x_n/a$.

Let us now consider case (ii) where the initial velocity is fixed ($V_0 > V_r$) and the amplitude of the external perturbation is ranged. Fig. 4(a) shows the behavior of $\overline{V}$ for different values of B_0 and for a fixed $V_0 = 50$. One sees that all curves start in a constant regime of initial velocity and after a crossover number n_x (as marked in the figure), they bend towards a regime of unlimited growth with the same slope. Such a growth is marked by a power law with exponent $\beta \approx 0.5$.

As discussed in Ref. [30], n is not a good variable but rather nB_0^2 is the ideal. When it is considered, an overlap for the curves of $\overline{V}$ is observed as shown in Fig. 4(b). Indeed considering the behavior of $\overline{V}$ curves shown in Figs. 3 and 4, we propose the following:

- (i) $\overline{V} \propto V_0^{\gamma}$, for $n \ll n_x$, where $\gamma = 1$ as one sees from Figs. 3 and 4;
- (ii) $\overline{V} \propto n^{\beta}$, for $n \gg n_x$, where $\beta \cong 0.5$;
- (iii) $n_x \propto V_0^{z_1} B_0^{y+z_2}$, where z_1 and z_2 are dynamical exponents as shown in Fig. 5 and $y = 2$, which comes from the change of variables describing only the ranging of B_0 as shown in Fig. 4.

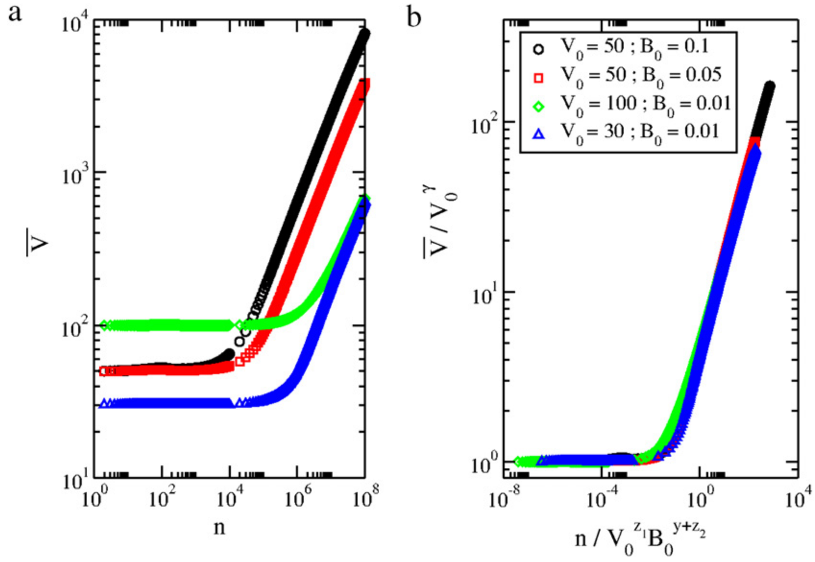

Fig. 6. (Color online) (a) Plot of $\overline{V} vs.n$ for different combinations of the control parameters initial velocity V_0 and amplitude of external boundary perturbation B_0. (b) Overlap of all curves of (a) into a single.

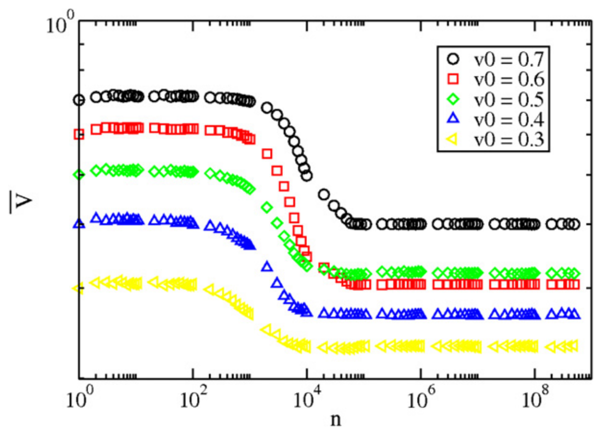

Fig. 7. (Color online) Plot of $\bar{V}$ vs $t(\mathrm{mod}\ 2\pi)$ for an ensemble of 5000 different initial conditions and different values of initial velocity. The decreasing on the velocity is evident.

The exponents z_1 and z_2 can be obtained from a plot of $n_x vs.V_0$ and $n_x vs.B_0$ as shown in Fig. 5. A power law fitting furnishes that $z_1 = 1.85(3)$ and $z_2 = -2.48(2)$.

Considering these three scaling suppositions we can now make an appropriate arrange in the axis to overlap all curves of $\overline{V}$ into a single plot as shown in Fig. 6. Therefore we can conclude that the average velocity, when initial condition is considered above the resonant one is scaling invariant with respect to the initial velocity V_0 and the amplitude of the external boundary perturbation (B_0).

We now discuss the case when the initial velocity is lower than the critical resonant one. We considered the same process as done before. We investigate the average velocity for an ensemble of 5000 different initial conditions evaluated until 5×10^8 collisions with the moving boundary. As shown in Fig. 7, a decreasing in the average velocity is observed for different values of initial velocity, (always lower than V_r). The curves of $\overline{V}$ stay constant during many collisions when suddenly they experience a crossover and decay for lower values. Given that the particle velocity in the ensemble tends to a constant value, we may conclude that FA is not taking place for this regime. The separation of the ensemble of particles by their velocity, where a part of them exhibits FA and part shows retardation, is a clear evidence of a Maxwell's Demon realization. One can therefore claim that the trapping in such a regime is due to stickiness which is still an open question and must be proved.

4. Final remarks and conclusions

As a short summary, we have investigated the dynamics of a driven stadium-like billiard. A four dimensional nonlinear mapping was found and a resonant velocity was recovered. It was obtained as a resonance between external boundary perturbation and the rotation number around the fixed points of the unperturbed map. Considering the resonant velocity as a "watershed" of the ensemble of fast and low particles, we showed a peculiar Maxwell's Demon, where fast particles experience the unlimited energy growth of the FA phenomenon, and low particles indeed decelerate. Scaling laws were studied and obtained for the high energy regime and a single plot was obtained for the average velocity, showing that its behavior is scaling invariant with respect to V_0 and B_0 for the case of $(V_0 > V_r)$.

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ALPL and EDL regret the death of a good friend and fabulous researcher, Alexander Loskutov, on the 5th November 2011.

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